Golden Ratio

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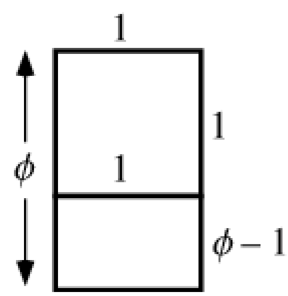
The golden ratio, also known as the divine proportion, golden mean, or golden section, is a number often encountered when taking the ratios of distances in simple geometric figures such as the pentagram, decagon and dodecagon. It is denoted ϕ, or sometimes τ (which is a abbreviation of the Greek "tome," meaning "to cut").

The term "golden section" (*goldene Schnitt*) seems to first have been used by Martin Ohm in the 1835 2nd edition of his textbook *Die Reine Elementar-Mathematik* (Livio 2002, p. 6). The first known use of this term in English is in James Sulley's 1875 article on aesthetics in the 9th edition of the *Encyclopedia Britannica.* The symbol ϕ ("phi") was apparently first used by Mark Barr at the beginning of the 20th century in commemoration of the Greek sculptor Phidias (ca. 490-430 BC), who a number of art historians claim made extensive use of the golden ratio in his works (Livio 2002, pp. 5-6).

ϕ has surprising connections with continued fractions and the Euclidean algorithm for computing the greatest common divisor of two integers. It is also a so-called Pisot Number.

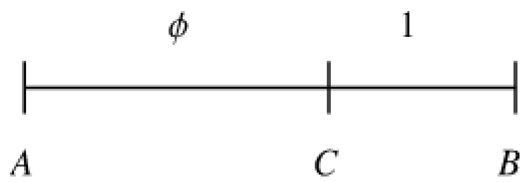


Given a rectangle having sides in the ratio $1:\phi$, ϕ is defined such that partitioning the origin rectangle into a square and new rectangle results in a new rectangle having sides with a rati $1:\phi$. Such a rectangle is called a golden rectangle, and successive points dividing a golden rectangle into squares lie on a logarithmic spiral. This figure is known as a whirling square.

The legs of a golden triangle (an isosceles triangle with a vertex angle of $36°$) are in a golde ratio to its base and, in fact, this was the method used by Pythagoras to construct ϕ. The rat of the circumradius to the length of the side of a decagon is also ϕ,

$$\frac{R}{s} = \frac{1}{2}\csc\left(\frac{\pi}{10}\right) = \frac{1}{2}\left(1+\sqrt{5}\right) = \phi. \quad ($$

Bisecting a (schematic) Gaullist cross also gives a golden ratio (Gardner 1961, p. 102).



Euclid ca. 300 BC defined the "extreme and mean ratios" on a line segment as the lengths such that

$$\phi = \frac{AC}{CB} = \frac{AB}{AC}$$ (

(Livio 2002, pp. 3-4). Plugging in,

$$\frac{\phi + 1}{\phi} = \phi,$$ (

and clearing denominators gives

$$\phi^2 - \phi - 1 = 0.$$ (

(Incidentally, this means that ϕ is a algebraic number of degree 2.) So, using the quadratic equation and taking the positive sign (since the figure is defined so that $\phi > 1$),

$$\begin{aligned} \phi &= \tfrac{1}{2}\left(1 + \sqrt{5}\right) \\ &= 1.618033988749894848204586834365638117720\ldots \end{aligned}$$ ((

(Sloane's A001622).

Exact trigonometric formulas for ϕ include

$$\begin{aligned} \phi &= 2\cos\left(\frac{\pi}{5}\right) \\ &= \frac{1}{2}\sec\left(\frac{2\pi}{5}\right) \\ &= \frac{1}{2}\csc\left(\frac{\pi}{10}\right). \end{aligned}$$ (((

The golden ratio is given by the infinite series

$$\phi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{(n+2)!\,n!\,4^{2n+3}}$$ (1

(B. Roselle). Another fascinating connection with the Fibonacci numbers is given by the infinite series

$$\phi = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n F_{n+1}}.$$ (1

A representation in terms of a nested radical is

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}} \quad (1$$

(Livio 2002, p. 83).

ϕ is the "most" irrational number because it has a continued fraction representation

$$\phi = [1, 1, 1, \ldots] = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} \quad (1$$

(Sloane's A000012; Williams 1979, p. 52; Steinhaus 1999, p. 45; Livio 2002, p. 84). This means that the convergents $x_n = p_n / q_n$ are given by the quadratic recurrence equation

$$x_n = 1 + \frac{1}{x_{n-1}}, \quad (1$$

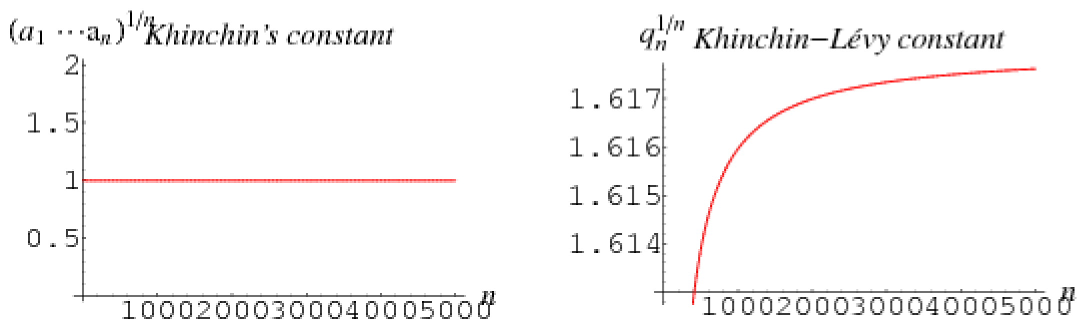
with $x_1 = 1$, which has solution

$$x_n = \frac{F_{n+1}}{F_n}, \quad (1$$

where F_n is the nth Fibonacci number. As a result,

$$\phi = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}, \quad (1$$

as first proved by Scottish mathematician Robert Simson in 1753 (Wells 1986, p. 62; Livio 2002, p. 101).



Let the continued fraction of ϕ be denoted $[a_0, a_1, a_2, \ldots]$ and let the denominators of the convergents be denoted q_1, q_2, ..., q_n. As can be seen from the plots above, the regularity i the continued fraction of ϕ means that ϕ is one of a set of numbers of measure 0 whose continued fraction sequences *do not* converge to the Khinchin constant or the Khinchin-Lévy constant.

The golden ratio has Engel expansion 1, 2, 5, 6, 13, 16, 16, 38, 48, 58, 104, ... (Sloane's A028259).

The golden ratio also satisfies the recurrence relation

$$\phi^n = \phi^{n-1} + \phi^{n-2}. \quad (1$$

Taking $n = 1$ gives the special case

$$\phi = \phi^{-1} + 1. \quad (1$$

Treating (◇) as a linear recurrence equation

$$\phi(n) = \phi(n-1) + \phi(n-2) \quad (1$$

in $\phi(n) = \phi^n$, setting $\phi(0) = 1$ and $\phi(1) = \phi$, and solving gives

$$\phi(n) = \phi^n, \quad (2$$

as expected. The powers of the golden ratio also satisfy

$$\phi^n = F_n \phi + F_{n-1}, \quad (2$$

where F_n is a Fibonacci number (Wells 1986, p. 39).

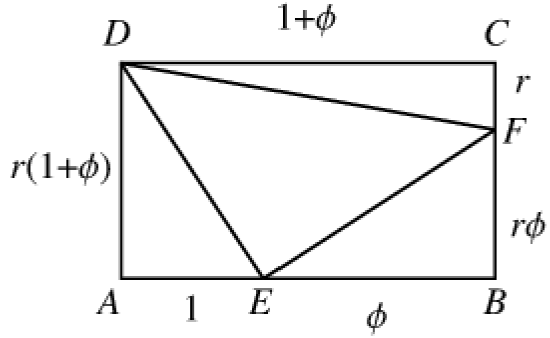
The sine of certain complex numbers involving ϕ gives particularly simple answers, for example

$$\sin(i \ln \phi) = \tfrac{1}{2} i \quad (2$$
$$\sin(\tfrac{1}{2}\pi - i \ln \phi) = \tfrac{1}{2}\sqrt{5} \quad (2$$

(D. Hoey, pers. comm.). A curious (although not particularly useful) approximation due to D Barron is given by

$$\phi \approx \tfrac{1}{2} K^{\gamma - 19/7} \pi^{2/7 + \gamma}, \quad (2$$

where K is Catalan's constant and γ is the Euler-Mascheroni constant, which is good to two digits.



In the figure above, three triangles can be inscribed in the rectangle $ABCD$ of arbitrary aspect ratio $1:r$ such that the three right triangles have equal areas by dividing AB and BC in the golden ratio. Then

$$K_{\Delta ADE} = \tfrac{1}{2} \cdot r(1+\phi) \cdot 1 = \tfrac{1}{2} r \phi^2 \quad (2$$
$$K_{\Delta BEF} = \tfrac{1}{2} \cdot r\phi \cdot \phi = \tfrac{1}{2} r \phi^2 \quad (2$$
$$K_{\Delta CDF} = \tfrac{1}{2}(1+\phi) \cdot r = \tfrac{1}{2} r \phi^2, \quad (2$$

which are all equal.

The substitution map

$$0 \rightarrow 01 \quad (2$$
$$1 \rightarrow 0 \quad (2$$

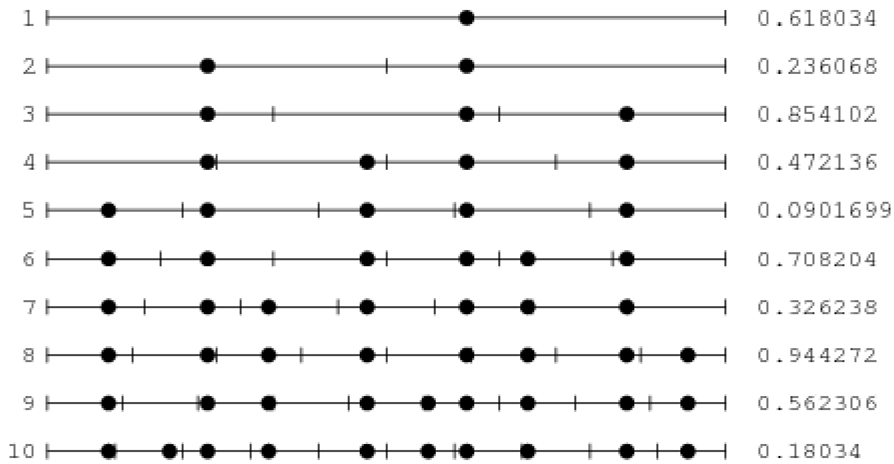
gives

$$0 \rightarrow 01 \rightarrow 010 \rightarrow 01001 \rightarrow \ldots, \quad (3$$

giving rise to the sequence

$$0100101001001010010100100101 \ldots \quad (3$$

(Sloane's A003849). Here, the zeros occur at positions 1, 3, 4, 6, 8, 9, 11, 12, ... (Sloane's A000201), and the ones occur at positions 2, 5, 7, 10, 13, 15, 18, ... (Sloane's A001950). These are complementary Beatty sequences generated by $\lfloor n\phi \rfloor$ and $\lfloor n\phi^2 \rfloor$. The sequence als has many connections with the Fibonacci numbers.



Steinhaus (1983, pp. 48-49) considers the distribution of the fractional parts of $n\phi$ in the intervals bounded by 0, $1/n$, $2/n$, ..., $(n-1)/n$, 1, and notes that they are much more uniformly distributed than would be expected due to chance (i.e., $\text{frac}(n\phi)$ is close to an equidistributed sequence). In particular, the number of empty intervals for $n = 1$, 2, ..., are mere 0, 0, 0, 0, 0, 0, 1, 0, 2, 0, 1, 1, 0, 2, 2, ... (Sloane's A036414). The values of n for which *no* bins are left blank are then given by 1, 2, 3, 4, 5, 6, 8, 10, 13, 16, 21, 34, 55, 89, 144, ... (Sloane's A036415). Steinhaus (1983) remarks that the highly uniform distribution has its roots in the continued fraction for ϕ.

The sequence $\{\text{frac}(x^n)\}$, of power fractional parts, where $\text{frac}(x)$ is the fractional part, is equidistributed for *almost all* real numbers $x > 1$, with the golden ratio being one exception.

Salem showed that the set of Pisot numbers is closed, with ϕ the smallest accumulation poin of the set (Le Lionnais 1983).

SEE ALSO: Beraha Constants, Decagon, Equidistributed Sequence, Euclidean Algorithm, Five Disks Problem, Golden Angle, Golden Gnomon, Golden Ratio Conjugate, Golden Rectangle, Golden Triangle, Icosidodecahedron, Noble Number, Pentagon, Pentagram, Phi Number System, Phyllotaxis, Pisot Number, Power Fractional Parts, Ramanujan Continued Fractions, Rogers-Ramanujan Continued Fraction, Secant Method. [Pages Linking Here]

RELATED WOLFRAM SITES: http://functions.wolfram.com/Constants/GoldenRatio/

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